HARTOGS' THEOREM ON SEPARATE HOLOMORPHICITY FOR PROJECTIVE SPACES

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Abstract. If a mapping of several complex variables into projective space is holomorphic in each pair of variables, then it is globally holomorphic.

1. Introduction

Hartogs' Theorem [2] on separately holomorphic functions is not valid for holomorphic mappings into a general complex manifold. Some basic counterexamples involve the complex projective space $\mathbb{P}^1$. Let $g : \mathbb{C}^2 \to \mathbb{P}^1$ be the mapping given by $g(x, y) = [xy, x^2 + y^2]$, with $g(0, 0) = [0, 1]$, in the homogeneous coordinates of $\mathbb{P}^1$. We have that g is a mapping holomorphic in each entry separately, but g is not even continuous at the origin. There are several works classifying those complex spaces for which the Hartogs' Theorem on separately holomorphic mappings holds. A short list includes [3], [4] and [5].

No complex projective space $\mathbb{P}^m$ satisfies the hypotheses presented in [3], [4] or [5]. The main objective of this work is to prove that a weak version of the Hartogs' Theorem holds for mappings into complex projective spaces $\mathbb{P}^m$.

By a coordinate k-plane Π^k, we understand any affine linear subspace of $\mathbb{C}^n$, with $n \geq k$, obtained by fixing $n - k$ of the coordinates. The original Hartogs' theorem may be stated as follows: given an open subset $\Omega \subset \mathbb{C}^n$, and a function $f : \Omega \to \mathbb{C}$ whose restriction to each intersection $\Omega \cap \Pi^1$ is holomorphic, for every coordinate 1-plane Π^1, we have that f is holomorphic on Ω. We may now present the main result of this paper.

Theorem 1 (Main). *Let Ω be an open domain in $\mathbb{C}^n$, with $n \geq 3$, and $\mathbb{P}^m$ be a complex projective space. Given a mapping $f : \Omega \to \mathbb{P}^m$ whose restriction to each intersection $\Omega \cap \Pi^2$ is holomorphic, for every coordinate 2-plane Π^2; we have that f is holomorphic on Ω.*

This theorem will be proved in the next section. The Main Theorem does not hold if we use continuity (or smoothness) instead of holomorphicity. For example, consider the mapping $h : \mathbb{C}^3 \to \mathbb{P}^1$ defined by $h(x, y, z) = [xyz, |x|^3+|y|^3+|z|^3]$, with $h(0, 0, 0) = [0, 1]$. The restriction of h to every coordinate 2-plane Π^2 is smooth, but h is not even continuous at the origin.

2. Proof of main theorem

We begin by recalling the following theorem of Alexander, Taylor and Ullman.

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Theorem 2. [1, p. 340] *Let Ω be an open subset of $\mathbb{C}^{n+1}$. Let $r \geq 1$ be a fixed integer, and $A \subset \Omega$ be a subset such that the intersection $A \cap \Pi^n$ with every coordinate n-plane Π^n is either empty or a subvariety of pure dimension r. Then, the set A is closed in Ω.*

Remark. The theorem in [1, p. 340] does not say explicitly that the intersection $A \cap \Pi^n$ may be empty. However, in their proof, they use the hypothesis that each set W_j is a r-dimensional subvariety of Ω_1, recalling their own notation. Looking carefully at their proof, we have that no set W_j can be empty, for the point c_j is indeed contained in W_j. Hence, the proof in [1, p. 340] works perfectly under the hypothesis that each intersection $A \cap \Pi^n$ is either empty or a r-dimensional subvariety.

Lemma 3. *Let Ω be an open domain in $\mathbb{C}^{n+1}$, with $n \geq 2$, and $\mathbb{P}^m$ be a complex projective space. Given a function $f : \Omega \to \mathbb{P}^m$ whose restriction to each intersection $\Omega \cap \Pi^n$ is holomorphic, for every coordinate n-plane Π^n, we have that f is holomorphic on Ω.*

Proof. We shall prove that f is holomorphic on a neighbourhood of any fixed point $z_0 \in \Omega$. Thus, we shall suppose from now on that Ω is a polydisc, and that $f(z_0) = [1, 0, 0, \ldots]$ in the homogeneous coordinates of $\mathbb{P}^m$. Consider the open set $U_0 \subset \mathbb{P}^m$ composed of the points $[1, \xi]$, for $\xi \in \mathbb{C}^m$. We only need to show that $f^{-1}(U_0)$ is an open neighbourhood of z_0 in Ω. Then, a direct application of the original Hartogs' theorem yields that f is holomorphic on $f^{-1}(U_0)$, because U_0 is obviously biholomorphic to $\mathbb{C}^m$; and so we have that f is holomorphic on a neighbourhood of z_0.

Notice that the hyperplane at infinity $\mathbb{P}^m \setminus U_0$ is the set of all points $[0, \xi]$ with $\xi \neq 0$. Define the set E equal to $f^{-1}(\mathbb{P}^m \setminus U_0)$. We assert that $E \cap \Pi^n$ is a subvariety of $\Omega \cap \Pi^n$, for every coordinate n-plane Π^n. Let z_1 be any given point in $E \cap \Pi^n$. We obviously have that $f(z_1) = [0, \xi]$ with $\xi \neq 0$. We can suppose, without loss of generality, that its second entry is different from zero. That is, $f(z_1) = [0, 1, \eta]$ for some $\eta \in \mathbb{C}^{m-1}$. Consider the open set $U_1 \subset \mathbb{P}^m$ composed of the points $[x, 1, y]$, with $x \in \mathbb{C}$ and $y \in \mathbb{C}^{m-1}$. Notice that $E \cap \Pi^n$ is a closed subset of $\Omega \cap \Pi^n$, and $f^{-1}(U_1) \cap \Pi^n$ is an open neighbourhood of z_1 in $\Omega \cap \Pi^n$ as well, because the restriction of f to $\Omega \cap \Pi^n$ is holomorphic. Besides, consider the holomorphic function $\pi : U_1 \to \mathbb{C}$, defined by $\pi[x, 1, y] = x$. We have that $\pi^{-1}(0)$ is equal to $(\mathbb{P}^m \setminus U_0) \cap U_1$. Hence, the set $E \cap \Pi^n$ is a subvariety of $\Omega \cap \Pi^n$ around z_1, because $E \cap \Pi^n \cap f^{-1}(U_1)$ is equal to the inverse image of zero under the holomorphic function $\pi \circ f|_{\Pi^n}$.

It is easy to deduce that $E \cap \Pi^n$ has only three possibilities: it can either be empty, equal to $\Omega \cap \Pi^n$, or a subvariety of $\Omega \cap \Pi^n$ with pure dimension $n-1$. Define J to be the union of all intersections $E \cap \Pi^n$ which are equal to $\Omega \cap \Pi^n$. We assert that J is a closed subset of Ω. Firstly, we consider only the coordinate n-planes whose first coordinate is constant, that is, planes $\Pi^{n,0}$ of the form $\{x\} \times \mathbb{C}^n$. Recall that Ω is a polydisc, so it can be written as the product $D_0 \times \Delta_0$, with D_0 an open disc in $\mathbb{C}$. Define $\rho_0 : \Omega \to D_0$ to be the projection on the first coordinate, and J_0 to be the union of all sets $E \cap \Pi^{n,0}$ which are equal to $\Omega \cap \Pi^{n,0}$. It is easy to deduce that

$$J_0 = \bigcap_{y \in \Delta_0} \rho_0(E \cap (D_0 \times \{y\})) \times \Delta_0.$$

Let H^n be any coordinate n-plane which contains the line $\mathbb{C} \times \{y\}$. We know that $E \cap H^n$ is a closed subset of $\Omega \cap H^n$. Whence, we also deduce that every $E \cap H^n$, each $E \cap (D_0 \times \{y\})$, and J_0 are all closed subsets of Ω. We may analyse, in the same way, the coordinate n-planes $\Pi^{n,k}$ of the form $\mathbb{C}^k \times \{x\} \times \mathbb{C}^{n-k}$, for $0 \leq k \leq n$, and define J_k to be union of all sets $E \cap \Pi^{n,k}$ which are equal to $\Omega \cap \Pi^{n,k}$. At the end, we deduce that each J_k is closed in Ω. Moreover, it is easy to deduce that $J = \bigcup_k J_k$ is also closed in Ω.

Finally, consider $E^* := E \setminus J$ and the open set $\Omega^* := \Omega \setminus J$ in $\mathbb{C}^{n+1}$. Every intersection $E^* \cap \Pi^n$ with a coordinate n-plane Π^n is either empty or an analytic set of pure dimension $n-1$. Then E^* is a closed subset of Ω^*, after applying Theorem 2. The sets $f^{-1}(U_0)$, $\Omega \setminus E$ and $\Omega^* \setminus E^*$ are all equal, so f is indeed holomorphic on the open neighbourhood $f^{-1}(U_0)$ of z_0; as we wanted to prove. □

The proof of the Main Theorem is a direct application of Lemma 3.

Proof. **(Main Theorem).** Considering Lemma 3, we may deduce that the restriction of f to each intersection $\Omega \cap \Pi^3$ is holomorphic, for every coordinate 3-plane Π^3. Proceeding by induction, we only need to apply Lemma 3 a finite number of times in order to deduce that the restriction of f to each intersection $\Omega \cap \Pi^k$ is holomorphic, for every coordinate k-plane Π^k, with $k \geq 3$; and so f if holomorphic on Ω. □

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